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by W. Lakin and L. Wolfenstein

Technical Report

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INTRODUCTION:

The spin state of a particle (or nucleus) taking part in a nuclear reaction generally must be described as a statistical mixture of the pure spin states in which the particle may be found. If the description consists of weighting equally all members of any basis set of mutually orthogonal spin functions, the mixture is spatially isotropic and describes unpolarized particles. Any different distribution will describe anisotropic states, and refer to polarized particles.(1) States which may be described by a single wave function rather than a more complicated statistical mixture will be called completely polarized. Except in the case of a particle of spin ½ the definition of the amount of polarization is ambiguous.

In Sec. I we consider ways of describing the polarization of particles of spin one, such as the deuteron. In Sec. II we determine the restrictions which the usual invariance requirements place upon the spin dependence of transition amplitudes. These sections are purely formal and are not dependent on any theory of nuclear interactions. In Sec. III we study the polarization of the deuterons produced in the $p + p \longrightarrow \pi^{+} + d$ reaction.(2) No detailed assumptions concerning the reaction are made, but limitations are put on the orbital angular momenta which should be valid if the energy is not too high. The use of elastic d-p scattering has been suggested as a possible means of detection of this polarization,(2) and this possibility is explored in Sec. IV. The results of such experiments depend upon the nature of the high energy scattering as well as the polarization of the deuterons. Although high energy nuclear scattering is not understood well, positive results in such experiments would provide qualitative evidence for the existence of polarization.

I. POLARIZATION STATES OF THE DEUTERON

The von Neumann density matrix ρ in spin space is often a convenient starting point in discussing polarization.(3) It may be expressed as a linear combination of

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(1) The term polarization here includes both polarization and alignment in the sense of Bleaney: B. Bleaney, Proc. Phys. Soc. (London) A64, 315 (1951); A. Simon, M. E. Rose, and J. M. Jauch, Phys. Rev. 84, 1155 (1951). Polarization and alignment are special cases of the vector and tensor types of polarization referred to later.

(2) Watson and Richman, Phys. Rev. 83, 1256 (1951). These authors discuss only that part of the polarization which may be specified by the expectation value of the spin.

(3) Wolfenstein and Ashkin, Phys. Rev. 85, 947 (1952) R.H. Dalitz, Proc. Phys. Soc. Lond. A65. 175 (1952). Our methods and notation generally follows that of the former paper.

independent hermitian matrices, whose number equals the square of the dimensionality of the spin space, and whose expectation values determine ρ and with it the state of the system. One of these matrices may be the identity matrix, the expectation value of which specifies the normalisation; that is, the trace of ρ. The remaining expectation values determine the polarization of the system.

We shall choose operators which form components of irreducible tensors[4] of rank zero (the identity matrix), one, and two. The operator which transforms under rotations like the spherical harmonic Y_{JM} is denoted by T_{JM}. This set is clearly equivalent to a set of independent hermitian operators, namely that set formed by the hermitian and antihermitian parts of these operators. The orthogonality and normalisation conditions are

$$\mathrm{Tr}\; T_{JM}\, T_{J'M'} = 3\,\delta_{JM,J'M'} \tag{1.1}$$

In terms of components of the deuteron spin operator $\underline{S}$, the T_{JM} are

$$\begin{aligned}
T_{1,-1} &= \tfrac{1}{2}\sqrt{3}\,(S_x - iS_y)\\
T_{1,0} &= \sqrt{\tfrac{3}{2}}\,S_z\\
T_{2,-2} &= \tfrac{1}{2}\sqrt{3}\,(S_x - iS_y)^2\\
T_{2,-1} &= \tfrac{1}{2}\sqrt{3}\,\{S_z(S_x - iS_y) + (S_x - iS_y)S_z\}\\
T_{2,0} &= \sqrt{\tfrac{1}{2}}\,(3S_z^2 - 2)\\
T_{J,M} &= (-1)^M\, T^{\dagger}_{J,-M}
\end{aligned} \tag{1.2}$$

The matrix elements of T_{JM} are determined completely by (1.1) and their tensorial properties

$$(m'|\,T_{JM}\,|m) = \sqrt{3}\;C_{1,J}^{-m',m,-M} \tag{1.3}$$

where $C_{j_1 j_2}^{m_1 m_2 M}$ is the usual Clebsch-Gordon coefficient for combining angular momenta j_1, j_2.

[4] See, for example, C. Racah, Phys. Rev. 62, 438 (1942)

The density matrix may now be written

$$\rho = \frac{1}{3} \sum_{JM} \langle T_{JM} \rangle \, T_{JM}^{\dagger} \tag{1.4}$$

where angular brackets denote statistical expectation values. The elements of ρ are restricted by the three independent Schwarz inequalities.

$$|\rho_{mm'nn'}|^2 \leq \rho_{mm'mm'} \, \rho_{nn'nn} \tag{1.5}$$

But for our purposes these are not very useful. A useful inequality (not independent of (1.5)) is the obvious one

$$\mathrm{Tr}\, \rho^2 \leq 1$$

$$\therefore \sum_{JM}{}' \; |\langle T_{JM} \rangle|^2 \leq 2 \tag{1.6}$$

where Σ' means that $J = 0$ (the identity) is to be omitted. This inequality is optimal in that the equality holds only in case of complete polarization, and the left hand side equals zero only for the unpolarized state. Therefore, one might define a percentage polarization by

$$P = \sqrt{\frac{1}{2} \sum_{JM}{}' \; |\langle T_{JM} \rangle|^2} \tag{1.7}$$

The components of $\underline{S}$ in general define a pseudovector whose direction may be used as the axis of quantization, so the $\langle T_{1,\pm 1} \rangle$ vanish. In Sec. II we shall show that in all states which may be produced in the simple reactions of interest to us, the $\langle T_{2,\pm 1} \rangle$ vanish too. We now restrict our consideration to this class of states and study methods of describing their state of

polarisation. Since rotating the frame of reference about the axis of quantisation alters the phase of the $\langle T_{2,\pm 2} \rangle$, we may so choose the coordinate system that these quantities are real and therefore equal (from (1.2)).

To study these states geometrically, we consider the cartesian components of the tensor operators, as given by (1.2). The expectation values of the components of the second rank tensor operator form the components of a symmetrical second rank tensor, which in turn defines a set of principal axes. Since $\langle T_{2,\pm 1} \rangle$ and $\langle T_{1,\pm 1} \rangle$ vanish, we see the $\hat{e}$-axis is the direction of the spin and also a principal axis of the second rank tensor. Geometrically the class of states we are studying is characterised by the fact that the direction of the spin is a principal axis of the tensor. We include in this class all of the singular cases in which it is not necessary but it is possible to have the spin direction parallel to a principal axis. These are the cases in which the expectation value of the spin vanishes, and the cases in which the principal axes of the tensor are not determined uniquely. The class of states now includes unpolarised particles.

The class of states in question includes all completely polarised states. To see this, we examine the wave function.

$$\psi = \alpha \chi_1 + \beta \chi_0 + \gamma \chi_{-1} \tag{1.8}$$

Here χ_m is an eigenvector of S_z belonging to the eigenvalue m. Since we have an arbitrary phase factor, we assume β is real. The matrices $\langle T_{1,-1} \rangle$ and $\langle T_{2,-1} \rangle$ are

$$T_{1,-1} = \sqrt{\tfrac{3}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T_{2,-1} = \sqrt{\tfrac{3}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \tag{1.9}$$

Therefore $\beta\alpha^* + \gamma^* \beta = \beta(\alpha + \gamma^*) = 0$ (1.10)

since $\langle T_{1,-1} \rangle$ vanishes because the z-axis has been chosen to be along the direction of the spin. Either β vanishes and with it $\langle T_{2,-1} \rangle$, or α

equals $|r|$ and then $\langle S_z \rangle$ vanishes.

Another method of specifying the polarisation state of a system is to tell which pure spin states are contained in the statistical mixture and the relative amounts of each state. We may start by noting that ρ is hermitian and may be diagonalised by a unitary transformation. Therefore the statistical mixture consists of the three orthogonal states which form the basis of this representation, and the corresponding eigenvalues of the density matrix are the relative amounts of these states. With the previous choice for the S_z-axis, $\langle T_{1,-1} \rangle$ and $\langle T_{2,-1} \rangle$ vanish, so that ρ must have the form (using Eq.(1.9)):

$$\rho = \begin{pmatrix} a & 0 & -c \\ 0 & 1-(a+b) & 0 \\ -c & 0 & b \end{pmatrix} \tag{1.11}$$

We assume temporarily that $a \neq b$, $c \neq 0$; these assumptions depend on choice of coordinates only and imply no loss of generality. (The use of a non-real quantity in place of c also means a change of coordinates).

Diagonalising ρ, we obtain the eigenvalues λ_i and their eigenvectors ψ_i.

$$\begin{aligned} \lambda_1 &= 1-(a+b) \\ \lambda_2 &= \tfrac{1}{2}\left[(a+b) + \sqrt{(a-b)^2+4c^2}\right] \\ \lambda_3 &= \tfrac{1}{2}\left[(a+b) - \sqrt{(a-b)^2+4c^2}\right] \end{aligned} \tag{1.12}$$

$$\begin{aligned} \psi_1 &= \chi_0 \\ \psi_2 &= (\lambda_2-\lambda_3)^{-\frac{1}{2}}\left[\sqrt{\lambda_2-b}\;\chi_{+1} - \sqrt{\lambda_2-a}\;\chi_{-1}\right] \\ \psi_3 &= (\lambda_2-\lambda_3)^{-\frac{1}{2}}\left[\sqrt{\lambda_2-a}\;\chi_{+1} + \sqrt{\lambda_2-b}\;\chi_{-1}\right] \end{aligned} \tag{1.13}$$

For degenerate eigenvalues, of course, the eigenfunctions are not unique. In the special case that the expectation value of the spin vanishes, we see from (1.11) that $a = b$. Then we have for Ψ_2 and Ψ_3

$$\Psi_2 = \frac{1}{\sqrt{2}} (\chi_{+1} - \chi_{-1})$$
$$\Psi_3 = \frac{1}{\sqrt{2}} (\chi_{+1} + \chi_{-1}) \qquad (1.14)$$

Since x_{+1} transforms like $-(x + iy)$ and x_{-1} transforms like $(x + iy)$, Ψ_2 and Ψ_3 transform like $(-x)$ and $(-iy)$, respectively, which means they represent states like χ_0, but oriented along the x- and y- axes, respectively.

Thus our restricted set of polarization states may be considered as a statistical mixture of a restricted set of pure states. In general we have one state χ_0 oriented along the direction of the total spin (z -axis), and a pair of states which are linear combinations of the states $\chi_{\pm 1}$. The average spins of the members of this pair are equal in magnitude, but opposite in direction. They are analogous to two elliptically polarized states of light with opposite senses of rotation and crossed major axes. The limiting cases are the state which is a mixture of the three oriented states χ_0, χ_{+1}, and χ_{-1}; or the state of zero spin. In the latter case, the pure states are χ_0 states oriented along three mutually perpendicular axes, the principal axes of the second rank tensor.

For a given set of principal axes, all states of this category are specified by the three real numbers, $\langle T_{20} \rangle$, $\langle T_{10} \rangle$, $\sqrt{2} \langle T_{2,-2} \rangle$.[5] Using (1.2) we see that if the state is simply the pure state Ψ_1, $\langle T_{20} \rangle = -\sqrt{2}$; and if it is any statistical mixture of the pure states Ψ_2 and Ψ_3, $\langle T_{20} \rangle = +\frac{1}{2}\sqrt{2}$. Since the general state of this category contains all three of these pure states, $\langle T_{10} \rangle$ varies from $-\sqrt{2}$ to $+\frac{1}{2}\sqrt{2}$. The other two operators must satisfy (1.6) for any given $\langle T_{20} \rangle$, but we may replace (1.6) by a more stringent inequality here.

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(5) We want the hermitian part of $T_{2,-2}$, as the expectation value of the anti-hermitian part must vanish. Therefore we use $\sqrt{2}$ as a normalization factor.

To calculate these expectation values, we use equations (1.11) and (1.12) and the explicit form of the tensor operators in question

$$T_{1,0} = \sqrt{\tfrac{3}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$T_{2,0} = \sqrt{\tfrac{1}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (1.15)$$

$$T_{2,-2} = \sqrt{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\langle T_{2,0} \rangle = \tfrac{1}{2}\sqrt{2}\,[\,a - 2(1-a-b) + b\,]$$
$$= \tfrac{1}{2}\sqrt{2}\,(1 - 3\lambda_1) \qquad (1.16)$$

$$\langle T_{1,0} \rangle = \sqrt{\tfrac{3}{2}}\,(a-b) \qquad (1.17)$$

$$\sqrt{2}\,\langle T_{2,-2} \rangle = \sqrt{\tfrac{3}{2}}\,(-2c) \qquad (1.18)$$

$$\therefore \langle T_{1,0} \rangle^2 + \left(\sqrt{2}\,\langle T_{2,-2} \rangle\right)^2 = \tfrac{3}{2}\{(a-b)^2 + 4c^2\}$$
$$= \tfrac{3}{2}\,(\lambda_2 - \lambda_3)^2 \qquad (1.19)$$

$$(\lambda_2 - \lambda_3)^2 \leq \lambda_2^2 \leq (1-\lambda_1)^2$$

$$\therefore \langle T_{1,0} \rangle^2 + \left(\sqrt{2}\,\langle T_{2,-2} \rangle\right)^2 \leq \tfrac{1}{3}\left(\langle T_{1,0} \rangle + \sqrt{2}\right)^2 \qquad (1.20)$$

Thus we see that for this category of states, (1.20) is the optimal inequality. If we represent each possible state by a point in a space whose coordinates represent

$\langle T_{10} \rangle$, $\sqrt{2}$ Re $\langle T_{2,-2} \rangle$, and $\langle T_{20} \rangle$ respectively, we see these points fill the interior of a cone,[6] whose apex is at $\langle T_{20} \rangle = -\sqrt{2}$ on the $\langle T_{20} \rangle$ - axis, and whose base has a radius of $\sqrt{\frac{3}{2}}$ and is normal to this axis at $\langle T_{20} \rangle = +\frac{1}{2}\sqrt{2}$. The completely polarized states are those for which either $\lambda_1 = 1$ or $\lambda_2 = 1$. These are characterized by points at the apex or circumference of the base of the cone respectively. The pair of states Ψ_2 and Ψ_3 will be on opposite ends of a diameter of the base.

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(6) To obtain the complete cone rather than one quarter of it, we drop the restrictions $a \geq b$, $c \geq 0$ used above.

II. SELECTION RULES FOR SIMPLE REACTIONS

We consider only those reactions in which both initial and final states contain only two particles (nuclei). The states are specified by the momenta in the CM system (center of mass) and the polarisation. This polarisation state may be specified by a density matrix in the spin space of the system, which is the product space of the spin spaces of the two particles.

The amplitude for a given transition may be represented as a matrix M whose rows and columns are characterized by the quantum numbers of final and initial states respectively.[7] In general M is rectangular. The density matrices describing initial and final states, denoted by ρ_i and ρ_f respectively, are related by the equation

$$I\rho_f = M\rho_i M^\dagger \tag{2.1}$$

$$I = Tr\,(M\rho_i M^\dagger) \tag{2.2}$$

Here I is the differential cross-section for the reaction at the angle in question.

We may now express ρ_i and ρ_f as linear combinations of basic matrices R^ν and S^μ in the spin spaces of the initial and final systems respectively. These basic matrices may be direct products of operators which operate on the state vectors of the individual particles. We then obtain the equations

$$I\rho_f = \frac{1}{n_i}\sum_\nu \langle R^\nu\rangle_i\, M R^{\nu\dagger} M^\dagger \tag{2.3}$$

$$I\langle S^\mu\rangle_f = \frac{1}{n_i}\sum_\nu \langle R^\nu\rangle_i\, Tr\,(M R^{\nu\dagger} M^\dagger S^\mu) \tag{2.4}$$

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[7] This and the following concepts and equations are a straightforward extension of the corresponding ones in reference 3.

Here angular brackets denote statistical expectation values, and the basic operators are normalised so that

$$\begin{aligned} \mathrm{Tr}\left(R^{\mu} R^{\nu\dagger}\right) &= n_i\, \delta_{\mu\nu} \\ \mathrm{Tr}\left(S^{\mu} S^{\nu\dagger}\right) &= n_f\, \delta_{\mu\nu} \end{aligned} \qquad (2.5)$$

where n_i and n_f are the dimensionalities of initial and final spin spaces. If neither initial particle is polarised and one reaction product is a deuteron and the polarisation of the other is not observed, the only $\langle R^{\mu}\rangle_i$ unequal to zero is the expectation value of the identity, and the only S^{μ} of interest will be the direct products of the tensor operators T_{JM} in the deuteron spin space with the identity operator in the spin space of the other reaction product. Then (2.4) becomes

$$I_0 \langle T_{JM}\rangle_f = \frac{1}{n_i} \mathrm{Tr}\left(M M^{\dagger} T_{JM}\right) \qquad (2.6)$$

where I_0 denotes the cross-section for an unpolarised initial state

Since H must be invariant with respect to rotations and space reflections, $MM^{\dagger}$ must be similarly invariant. Now $MM^{\dagger}$ is a hermitian matrix in the spin space of the final system, and terms of interest must be contractions of the T_{JM} with tensor quantities formed from $\underline{k}_i$ and $\underline{k}_f$, the initial and final momenta. To be invariant under space reflection, the terms must be of even degree in $\underline{k}_i$ and $\underline{k}_f$, since the T_{JM} are invariant under space reflection. Consequently the most general form of $MM^{\dagger}$ is

$$\begin{aligned} MM^{\dagger} = {} & J_0 + J_1 \sum_M Y_{2M}(\hat{\underline{k}}_i)\, T_{2M}^{\dagger} + J_2 \sum_M Y_{2M}(\hat{\underline{k}}_f)\, T_{2M}^{\dagger} \\ & + J_3 \sum_M Y_{2M}(\hat{\underline{k}}_i, \hat{\underline{k}}_f)\, T_{2M}^{\dagger} + J_4 \sum_M Y_{1M}(\hat{\underline{k}}_i \times \hat{\underline{k}}_f)\, T_{1M}^{\dagger} \end{aligned} \qquad (2.7)$$

Here the quantities J_K depend only on $\underline{k}_i$, $\underline{k}_f$, $\underline{k}_-$, and k_f, and must be real in order that MM^+ is hermitian. The Y_{JM} are solid spherical harmonics formed from the components of their indicated arguments,[8] and + ... indicates quantities depending on operators in the spin space of the second reaction product.

Equations (2.6) and (2.7) show that $\langle T_{JM} \rangle_f$ equals the coefficient of T_{JM}^+ in (2.7) divided by I_0, the terms denoted by + ... contributing nothing to these. To study the polarization state of these deuterons, we choose the normal to the reaction plane as $\hat{y}$-axis, and use the following form of spherical harmonics:

$$
\begin{aligned}
Y_{1,-1}(\underline{1}) &= \sqrt{\tfrac{1}{2}}\,(x - iy) \\
Y_{1,0}(\underline{1}) &= z \\
Y_{2,-2}(\underline{1}) &= \sqrt{\tfrac{3}{8}}\,(x - iy)^2 \\
Y_{2,-1}(\underline{1}) &= \sqrt{\tfrac{3}{2}}\,(x - iy)\,z \\
Y_{2,0}(\underline{1}) &= \tfrac{1}{2}\,(2z^2 - x^2 - y^2) \\
Y_{J,-M}(\underline{1}) &= (-1)^M\, Y_{J,M}^*(\underline{1})
\end{aligned}
\qquad (2.8)
$$

with the above choice of polar axis, the $\hat{y}$ - components of $\underline{k}_i$ and $\underline{k}_f$ vanish, and $\underline{k}_i \times \underline{k}_f$ has only a $\hat{y}$ - component. Therefore all harmonics of the type

[8] $Y_{2M}(\underline{k}_i, \underline{k}_f)$ is a second-degree harmonic, bilinear in $\underline{k}_i$ and $\underline{k}_f$, and symmetrical in these.

$Y_{J,\pm 1}$ [9] in (2.7) vanish, and with these all $\langle T_{J,\pm 1}\rangle_f$. Thus, our assertion in Sec. I that the polarization states produced in simple reactions will have the spin as a principal axis of the second rank tensor is proved, and moreover, we see this preferred direction is normal to the reaction plane.

We have already discussed the general features of these polarization states. It is useful to examine the $\langle T_{JM}\rangle_f$ using an axis in the reaction plane as the polar axis. If we try to detect polarization by a second reaction, it is convenient to choose the direction of incidence of this second reaction as β - axis, which is in the plane of the first reaction. However, we single out the normal by using it as the Y-axis. In this coordinate system, we see that all second rank harmonics from $\underline{k}_i$ and $\underline{k}_f$ in (2.7) are real, and $Y_{1,\pm 1}$ $(\underline{k}_i \times \underline{k}_f)$ are pure imaginary. Since these determine the $\langle T_{JM}\rangle_f$ as noted above, we obtain in conjunction with (1.2)

$$\langle T_{2M}\rangle_f = (-1)^M \langle T_{2,-M}\rangle_f \tag{2.9}$$

$$\langle T_{1M}\rangle_f = (-1)^{M+1} \langle T_{1,-M}\rangle_f \tag{2.10}$$

Equation (2.10) includes the vanishing of $\langle T_1\rangle_f$, which we know from the preceding analysis, since $\langle \underline{S}\rangle_f$ lies on the normal

A second problem of interest is the angular distribution of the reaction products, if the initial state consists of a deuteron, which may be arbitrarily polarized, and an unpolarized target of spin j. We assume there are only two reaction products again. We use equation (2.4) with the identity matrix as

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(9) It is clear that these arguments hold for particles with higher spin, and for particles with spin 1/2 (ref. 3)

8^{μ}, and direct products of the T_{JM} with the identity matrix in the spin space of the target as the R^{ν} of interest. We then have

$$I = \frac{1}{3(2J+1)} \sum_{M} \langle T_{JM} \rangle_i \, \mathrm{Tr} \left(M^{\dagger} M \, T_{JM}^{\dagger} \right) \qquad (2.11)$$

The hermitian matrix $M^{\dagger}M$ is an operator in the spin space of the initial system, and may be expressed as a linear combination of basic operators in that space. Since $M^{\dagger}M$ must be invariant under rotations and space reflections

$$M^{\dagger}M = I_0 + I_1 \sum_M \psi^*_{1M}(\underline{k}_i) T_{1M} + I_2 \sum_M \psi^*_{2M}(\underline{k}_f) T_{1M}$$
$$+ I_3 \sum_M \psi^*_{2M}(\underline{k}_i, \underline{k}_f) T_{1M} + I_4 \sum_M \psi^*_{1M}(\underline{k}_i \times \underline{k}_f) T_{1M}$$
$$+ \cdots \qquad (2.12)$$

As in (2.7) the I are real scalars depending on k_1, k_f, and $\underline{k}_i \cdot \underline{k}_f$ only, and $+\cdots$ indicates terms containing operators other than the identity in the spin space of the target.

To obtain the angular distribution of the reaction products, we substitute (2.12) into (2.11).

$$I = I_0 + I_1 \sum_M \psi^*_{1M}(\underline{k}_i) \langle T_{1M} \rangle_i + I_2 \sum_M \psi^*_{2M}(\underline{k}_f) \langle T_{1M} \rangle_i$$
$$+ I_3 \sum_M \psi^*_{2M}(\underline{k}_i, \underline{k}_f) \langle T_{1M} \rangle_i + I_4 \sum_M \psi^*_{1M}(\underline{k}_i \times \underline{k}_f) \langle T_{1M} \rangle_i$$

(2.13)

To study this in greater detail, we choose the polar axis parallel to $\underline{k}_i$. The angular distribution expresses the dependence of I upon $\underline{k}_f$. Since the only vector with non-vanishing X and Y components is $\underline{k}_f$, the harmonics which multiply $(T_{JM})_i$ in (2.13) may be expressed as a sum of terms containing $Y_{LM}(\underline{k}_f)$ (where the M's are the same) times quantities independent of the azimuthal angle. Therefore, the azimuthal angular distribution of the reaction products is directly related to the polarization of the incident deuterons. The polar angular distribution may also depend upon $\langle T_{JM} \rangle_i$, but generally this would not be related in any simple manner to the polarization of the incident particles.

We now consider the case in which the polarized deuterons used as projectiles in a reaction were produced in a previous simple reaction. We use the incident direction of the second reaction as the z axis and the plane of the first reaction as the xy - plane. Then the $\langle T_{JM} \rangle_i$ in equation (2.13) applied to the second reaction must satisfy (2.9) and (2.10). Combining the terms containing $Y^*_{JM}(\underline{k}_f)$ and $Y^*_{J-M}(\underline{k}_f)$, we see that the angular distribution will have the form

$$I(\theta,\phi) = I_0(\theta,\phi) + A(\theta)\langle T_{10}\rangle_i + \left(B(\theta)\langle T_{21}\rangle_i + C(\theta)\left|\langle T_{11}\rangle_i\right| \right) \sin\theta \cos\phi + D(\theta)\langle T_{22}\rangle_i \sin^2\theta \cos 2\phi \qquad (2.14)$$

Here θ is the polar angle, ϕ the azimuthal angle measured from the plane of the first reaction, and the functions I, A, B, C, D, are polynomials in $\cos\theta$.

If L_{max} is the maximum effective incident orbital angular momentum (in the second reaction), the degrees of these polynomials are limited as follows:[10]

polynomial	maximum degree in $\cos\theta$
$I_0(\theta)$	$? L_{max}$
$A(\theta)$	$2L_{max} + 2$
$B(\theta)$	$2L_{max} + 1$
$C(\theta)$	$2L_{max} - 1$
$D(\theta)$	$2L_{max}$

The above rules for polarization effects in the simpler nuclear reactions depend only upon the invariance of the Hamiltonian under rotations and reflections. In the special case of elastic scattering, a further restriction is imposed upon M if the Hamiltonian is also required to be invariant under time reversal.[11] Since the time-reverse collision is identical with the original, except for a pure rotation, the matrix elements for these must be identical. If the time reverse of M is defined by:

$$M'_{a,b} = M_{-b,-a} \tag{2.15}$$

where -b, -a refer to the time-reversed states,

$$M' = M \;;\; (MM^{\dagger})' = M^{\dagger}M \tag{2.16}$$

Using primes to denote time reversed quantities, we have:

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(10) The argument is an extension of that used for unpolarized beams; see, e.g., Blatt + Weisskopf, Theoretical Nuclear Physics, p 535

(11) E. Wigner, Gött Nachs. 31, 546 (1932)

$$\underline{k}_i' = -\underline{k}_f$$
$$\underline{k}_f' = -\underline{k}_i$$
$$\underline{S}' = -\underline{S} \qquad (2.17)$$

$$T_{JM}' = (-1)^J T_{JM}$$
$$Y_{JM}(\underline{k}_i') = (-1)^J Y_{JM}(\underline{k}_f)$$
$$Y_{JM}(\underline{k}_f') = (-1)^J Y_{JM}(\underline{k}_i') \qquad (2.18)$$

Substituting (2.18) into (2.7) and (2.12), we obtain directly from (2.16):

$$J_0 = I_0$$
$$J_1 = I_2$$
$$J_2 = I_1$$
$$J_3 = I_3$$
$$J_4 = I_4 \qquad (2.19)$$

As in ref. 3, these equations relate the polarisation produced in elastic collisions if the initial particles are unpolarized to the effect of polarization of the incident particle on the angular distribution of elastically scattered particles. These relations are of interest if double scatterings are studied and if the change in I_n due to energy loss in the first collision may be neglected (or is known)

III. Polarization of the Deuterons Produced In The Reaction $p + p \rightarrow \pi^+ + d$.

According to the customary assumption, the parity of the orbital motion of the system must change in a reaction in which a π-meson is produced. Also the states of relative motion of the final system are assumed to be only S and P states at the energy under consideration. These assumptions limit the form of H to the following:

$$H = \Lambda\left[A\,(\sigma_1+\sigma_2)\cdot k + B\,k\cdot q\,(\sigma_1-\sigma_2)\cdot l + C\,(\sigma_1\cdot\sigma_2)\,q\right] \tag{3.1}$$

Here Λ is the triplet projection operator; σ_1 and σ_2 are Pauli matrices operating on the spin functions of the two nucleons; k is the relative momentum in the initial state; q the relative momentum in the final state, so chosen that $+q$ represents the momentum of the deuteron; and A, B, and C depend only upon k. The terms which appear in H must be pseudoscalar to change the parity of the orbital motion. Also H must change sign if the two nucleons are exchanged, since it is a matrix element of an operator which must be symmetrical in the two nucleons, and the initial state is antisymmetrical and the final state symmetrical. Exchanging space and spin coordinates is equivalent to exchanging σ_1 and σ_2 and changing the sign of k. The only pseudoscalar operators satisfying this condition which can be constructed from k, q, σ_1, and σ_2, which lead to a deuteron in the final state, and which contain no power of q higher than the first are the terms in (3.1).

It may seem at first that the use of H as an operator in the two nucleon spin space implicitly assumes the ground state of the deuteron is a pure S-state. This is not the case, however, since the matrix elements of any operator connecting the deuteron ground state and the initial state may be represented as the matrix elements of a suitably modified operator in the

two-nucleon spin space. The deuteron wave function may be written [12]

$$\psi_m = \frac{1}{r}\left(u + w S_{12}\right) \chi_m \tag{3.2}$$

In evaluating matrix elements of any operator, we may transfer the part of ψ_m preceding χ_m to the operator in question. The only difference between assuming the deuteron to be in a pure S-state or taking the D-admixture into account appears in the parameters A, B, and C.

We now change parameters in (3.1), introducing unit vectors $\hat{k}$ and $\hat{q}$ parallel to $\underline{k}$ and $\underline{q}$ respectively

$$M = \Lambda \left[A\,(\underline{\sigma}_1 + \underline{\sigma}_2)\cdot\hat{k} + B\,\hat{k}\cdot\hat{q}\,(\underline{\sigma}_1 - \underline{\sigma}_2)\cdot\hat{k} + C\,(\underline{\sigma}_1 - \underline{\sigma}_2)\cdot\hat{q} \right] \tag{3.3}$$

$$\begin{aligned} MM^\dagger = \Lambda \Big[& \frac{4\pi}{3}\left\{(2|A|^2 + |C|^2) + (|B|^2 + 2\,\mathrm{Re}\,B^*C)\cos^2\theta\right\} \\ & + \frac{2}{3}\left\{|A|^2 - (|B|^2 + 2\,\mathrm{Re}\,B^*C)\cos^2\theta\right\}(3\underline{\sigma}_1\cdot\hat{k}\,\underline{\sigma}_2\cdot\hat{k} - 1) \\ & - \frac{2}{3}|C|^2\,(3\,\underline{\sigma}_1\cdot\hat{q}\,\underline{\sigma}_2\cdot\hat{q} - 1) \\ & - 2\,\mathrm{Re}\,B^*C\cos\theta\sin\theta\,(\underline{\sigma}_1\cdot\hat{k}\,\underline{\sigma}_2\cdot\hat{p} + \underline{\sigma}_1\cdot\hat{p}\,\underline{\sigma}_2\cdot\hat{k}) \\ & + 2\,\mathrm{Im}\,B^*C\cos\theta\sin\theta\,(\underline{\sigma}_1 + \underline{\sigma}_2)\cdot\hat{n} \Big] \end{aligned} \tag{3.4}$$

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[12] W. Rarita and J. S. Schwinger, Phys. Rev. 59, 436 (1941)

Here θ is the angle between $\hat{k}$ and $\hat{q}$, $\hat{p}$ is a unit vector in the plane of $\hat{k}$ and $\hat{q}$ perpendicular to $\hat{k}$, and $\hat{n}$ is a unit vector normal to the plane of $\hat{k}$ and $\hat{q}$.

The form of MM^+ may be simplified if the emerging deuterons are chosen from an angular range symmetrical about ninety degrees (CM), providing the variation of energy with angle in this range may be neglected. Then all terms proportional to $\sin\theta\cos\theta$ vanish, and MM^+ will not contain any first rank tensor operators.(13) This simplification is important because if the polarisation is to be detected by asimuthal asymmetries in a second reaction, it would in general be difficult to differentiate between asymmetries associated with $\langle T_2, t_1 \rangle$ and $\langle T_2, t_1 \rangle$. The feasibility of utilising this idea in any experiment depends on the angular distribution of the $p + p \rightarrow \pi^+ + d$ reaction and also on the energy dependence of the detecting reaction. For 400 Mev incident protons, the energy (lab) of the emerging deuterons will be

$$E = 190 \ (1 + 0.29 \cos\theta) \ \text{Mev} \qquad (3.5)$$

The allowable variation of energy then determines the range of θ, and the experiment will only be feasible if this range contains a sufficiently intense beam of outgoing deuterons. We assume for the remainder of this paper that the above method of eliminating all polarisation associated with first order tensors is used.

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(13) It may be seen that the expectation value of the spin vanishes even if higher partial waves are involved. Because of the exclusion principle in the initial state, even and odd states cannot interfere, and the intensity is a sum of even harmonics. The term containing the spin must be proportional to sin θ; therefore it must be multiplied by an odd polynomial in cos θ.

We now use the notation

$$a = \frac{2|A|^2 + |C|^2}{|B|^2 + 2\,\mathrm{Re}\,B^*C} \tag{3.6}$$

$$\epsilon = \frac{2|A|^2}{2|A|^2 + |C|^2} \tag{3.7}$$

We also use the identity

$$3\,\underline{\sigma}_1\cdot\hat{k}\;\underline{\sigma}_2\cdot\hat{k} - \underline{\sigma}_1\cdot\underline{\sigma}_2 = \sqrt{8}\sum_M Y_{2M}(\hat{k})\,T_{2M}^{\dagger} \tag{3.8}$$

Also a bar is used to denote quantities averaged over angles, and equation (3.4) becomes

$$\overline{MM^{\dagger}} = (\text{const})\Big[\Big(a + \overline{\cos^2\theta}\Big) + \Big(\tfrac{1}{2}\epsilon a - \overline{\cos^2\theta}\Big)\cdot\sqrt{2}\sum_M Y_{2M}(\hat{k})\,T_{2M}^{\dagger} - (1-\epsilon)a\sqrt{2}\sum_M \overline{Y_{2M}(\hat{q})}\;T_{2M}^{\dagger}\Big] \tag{3.9}$$

If we choose the polar axis along $\hat{k}$, $\overline{Y_{2\pm1}(\hat{q})}$ vanishes since they are proportional to $\sin\theta\cos\theta$. Then $\langle T_{2\pm1}\rangle$ vanishes as well as $\langle T_{2\pm1}\rangle$, and $\hat{k}$ is a principal axis as well as $\hat{n}$. Therefore, $\hat{k}$, $\hat{p}$, and $\hat{n}$ define the principal axes of the second rank tensor. In the extreme case where $\overline{\cos^2\theta}$ may be neglected, we obtain using (2.6)

$$\langle T_{20}\rangle = \frac{1}{2}\sqrt{2} \tag{3.10}$$

$$\langle T_{2,-2}\rangle = -\frac{1}{2}\sqrt{3}\,(1-\epsilon) \tag{3.11}$$

According to the discussion of Sec. I, the statistical mixture in this extreme case will contain states like χ_0 oriented along $\hat{p}$ and $\hat{n}$ only for all values of E. If E = 0[14] the state will be a pure state χ_0 oriented along $\hat{p}$; if E = 1 states oriented along $\hat{p}$ and $\hat{n}$ have equal weights. In less extreme cases where $\overline{\cos^2\theta}$ cannot be neglected the value of $\langle T_{20} \rangle$ will be less than $\frac{1}{2}\sqrt{2}$ and the statistical mixture will also contain the state oriented along $\hat{k}$.

To consider the detection of the polarisation by azimuthal asymmetries in a second reaction, it is convenient to use the direction of motion (lab system) as the S axis and the normal to the plane of the first reaction as the y-axis. The value of $\langle T_{2M} \rangle$ and also of P defined by (1.7) in this coordinate system for various a and ξ are given in Table II, where the incident proton energy is 400 Mev and $\overline{\cos^2\theta}$ equals 0.039 corresponding to the range $70^\circ \leqslant \theta \leqslant 110^\circ$. The values of "a" are chosen to approximate the reported angular distributions in the reaction $p + p \rightarrow \pi^+ + d$, or its inverse.[15][16] The magnitudes of $\langle T_{2,1} \rangle$ and $\langle T_{2,2} \rangle$ decrease with increasing E; the magnitude of $\langle T_{2,2} \rangle$ is almost proportional to $(1-\xi)$. For the values of "a" considered, the results are very sensitive to $\overline{\cos^2\theta}$ as well as E. The magnitude of the expectation values of all operators would increase if the angular range were decreased.

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[14] This for example is used by Chew et al, Phys. Rev. 84, 581 (1951)

[15] Whitehead and Richman, Phys. Rev. 83, 855 (1951)

[16] Durbin, Loar, and Steinberger, Phys. Rev. 84, 581 (1951)

Finally, we note that in case the angular distribution of this reaction is proportional to $\cos^2\theta$, the polarisation is complete and independent of angle, the statistical mixture consisting of the pure state oriented along $\hat{k}$. This may be readily seen by setting A and C equal to zero in (3.4). Since the angle of the emerging beam (lab system) would have to make an angle of less than eight degrees with $\hat{k}$, the expectation value of all tensor operators[17] except $\langle T_{20} \rangle$ would be small. If A and C were not zero but very small, so that it would be impossible to choose the emerging beam to be symmetrical about ninety degrees, the contributions of the last two terms in (3.4) would not be negligible, and the results for A and C vanishing would be a very poor approximation to this problem. For this reason we have not calculated the case where A and C vanish. In principle one could change the orientation of the states relative to the direction of motion by magnetic deflection. In this case small values of $|A|$ and $|C|$ would be of interest, since the polarisation would be almost complete and could be estimated accurately if the emerging deuterons are in the forward direction.

Table II

α	ϵ	$\langle T_{2,0} \rangle$	$\langle T_{2,1} \rangle$	$\langle T_{2,2} \rangle$	P
0.1	0.0	-0.021	+0.092	-0.59	0.60
	0.5	+0.044	+0.065	-0.30	0.30
	1.0	+0.11	+0.019	+0.0013	0.08
0.2	0.0	+0.21	+0.14	-0.70	0.73
	0.5	+0.28	+0.10	-0.34	0.42
	1.0	+0.36	+0.061	+0.042	0.26

(17) We again consider the direction of motion (lab system) as the z-axis and normal to plane of first reaction as the y-axis, since we are interested in possible detection by azimuthal asymmetries in a second reaction.

IV. EFFECTS OF DEUTERON POLARIZATION ON d-p SCATTERING

It has been suggested that the elastic scattering of deuterons by hydrogen might be used to detect the polarization of the deuterons.[2] Here the quantities denoted by I_q in (2.12) have a complicated dependence on angle[18], and the effects of polarization on polar angular distributions are difficult to interpret. Therefore we consider only the effects of polarization of the incident deuterons on the azimuthal angular distribution. As shown in Sec. II, any observed azimuthal asymmetry verifies the existence of the relevant polarization of the incident beam. In the problem we are considering, a left-right asymmetry (i.e. $\cos\phi$ term in the angular distribution) would be attributable to the $\langle T_{2\,2} \rangle_1$ terms, since the expectation value of the spin vanishes. The magnitude of these polarization effects is determined by the product of the expectation value of the relevant operator in the incident beam and the magnitude of the relevant term in the scattered intensity operator.

In this section we shall use the impulse approximation[19] to calculate the elastic d-p scattering amplitude in terms of the p-p and n-p scattering amplitudes. Each nucleon-nucleon scattering amplitude will be calculated on the basis of a model chosen primarily for its simplicity. Also various approximations in addition to the impulse approximation will be made. Therefore the final result is not expected to be good, but it may indicate the order of magnitude of the azimuthal asymmetries without excessive labor. We do not

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[18] Chew, Phys. Rev. 84, 710 (1951), and references in this paper.

[19] Chew, Phys. Rev. 80, 196 (1950).

believe a long calculation is warranted at present.

We wish to calculate a matrix H specifying the scattering, which operates in the product space of the deuteron and proton spin functions. Since we shall use the approximation that the deuteron is a pure S-state, the spin space to be used is equivalent to the product of the spin spaces of the three nucleons. It is convenient to use a representation in which the proton bound in the deuteron and the free proton are distinguished. This is often done by taking the exclusion principle into account in the matrix H rather than in the wave functions.

We designate the neutron as particle one, the proton bound in the deuteron as two, and the free proton as three. Initial and final state functions (not antisymmetrical in the two protons) are denoted by Ψ_i and Ψ_f respectively, and P_{23} and Q_{23} denote space (momentum) and spin exchange operators for the two protons. The properly antisymmetrized function representing initial and final states will then be $\frac{1}{2}\sqrt{2}\,(1-P_{23}Q_{23})\Psi_i$ and $\frac{1}{2}\sqrt{2}\,(1-P_{23}Q_{23})\Psi_f$ respectively. A straightforward calculation of the elastic scattering amplitude using these antisymmetrized functions and the impulse approximation yields the result

$$H = H_1 + H_2 + H_3 \tag{4.1}$$

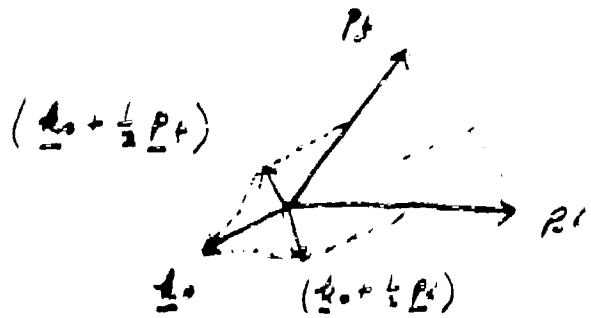
$$M_1 = \int d_3 k\; f\left(\underline{k}+\tfrac{1}{2}\underline{p}_f\right)\left(\tfrac{1}{2}\underline{k}+\underline{p}_f\,\middle|\,\{1-P_{23}Q_{23}\}R_{np}(12)\,\middle|\,\tfrac{1}{2}\underline{k}+\underline{p}_i\right) \cdot f\left(\underline{k}+\tfrac{1}{2}\underline{p}_i\right) \tag{4.2}$$

$$M_2 = \int d_3 k\; f\left(k+\tfrac{1}{2}p_f\right)\left(\tfrac{1}{2}\underline{k}+\underline{p}_f\,\middle|\,R_{np}(13)\,\middle|\,\tfrac{1}{2}\underline{k}+\underline{p}_i\right) f\left(\underline{k}+\tfrac{1}{2}\underline{p}_i\right) \tag{4.3}$$

$$M_3 = -Q_{23}\, f\left(p_f+\tfrac{1}{2}\underline{p}_i\right)\int d_3 k\; f\left(\underline{k}\right)\left(\underline{k}\,/\,R_{np}(13)\,\middle|\,\underline{p}_i+\tfrac{1}{2}\underline{p}_f\right) \tag{4.4}$$

Here $\underline{p}_i$ and $\underline{p}_f$ are the initial and final relative momenta, $R_{pp}(23)$ and $R_{np}(23)$ are scattering amplitudes for the p-p and n-p collisions between states with the indicated relative momenta, and f is the momentum wave function for the deuteron. Since the transitions indicated by the R-matrices in general do not conserve energy, their calculation from scattering data involves an appropriate extrapolation.

The fourier transform f of the deuteron wave function is peaked near the origin in the momentum space and is small for arguments whose magnitudes are large. Consequently, if the polar angles of scattering are restricted to the first quadrant and initial and final momenta are large compared to momenta of the nucleons inside the deuteron, $f(\underline{p}_f + \frac{1}{2}\underline{p}_i)$ can be assumed small and M_3 neglected. Furthermore, the peakedness of f may be used to approximate the expressions for M_1 and M_2. Since the arguments of both deuteron functions in M_1(or M_2) do not vanish simultaneously for any value of $\underline{k}$, we assume the peak of the product is at the value of $\underline{k}$ for which both arguments have equal magnitudes, this magnitude being the minimum consistent with this restriction. We assume the nucleon-nucleon scattering amplitudes may be taken out of the integral at this value of $\underline{k}$.



Denoting this value of $\underline{k}$ by $\underline{k}_0$, we see from the diagram that $\underline{k}_0$ equals $-\frac{1}{2}(\underline{p}_f + \underline{p}_i)$.

With this approximation, equations (4.2) and (4.3) become

$$M_1 = \left(\tfrac{1}{2}\underline{k}_0 + \underline{p}_f \middle| \{1 - P_{23} Q_{23}\} R_{pp}(23) \middle| \tfrac{1}{2}\underline{k}_0 + \underline{p}_i \right) F(\underline{p}_f - \underline{p}_i) \qquad (4.5)$$

$$M_2 = \left(\tfrac{1}{2}\underline{k}_0 + \underline{p}_f \middle| R_{np}(13) \middle| \tfrac{1}{2}\underline{k}_0 + \underline{p}_i \right) F(\underline{p}_f - \underline{p}_i) \qquad (4.6)$$

$$F(\underline{p}_f - \underline{p}_i) = \int d_3 k \, f\left(\underline{k} + \tfrac{1}{2}\underline{p}_f\right) f\left(\underline{k} + \tfrac{1}{2}\underline{p}_i\right)$$

$$= \frac{1}{(2\pi)^3} \int e^{-\frac{1}{2} i (\underline{p}_f - \underline{p}_i)\cdot \underline{r}} \, \varphi(r)^2 \, d_3 r \qquad (4.7)$$

Here $\varphi(\underline{r})$ is the deuteron wave function in coordinate space, assuming a pure S-function.

The quantity $|\underline{k}_0 + \underline{p}_i|$ depends on the scattering angle. This angle must be in the first quadrant, and according to (2.14) it should not be too close to zero. We have chosen sixty degrees and thirty degrees as scattering angles. The energies (lab system) of nucleon-nucleon scattering which determine the R-matrices in (4.5) are 113 and 100 mev respectively. At these energies, the p-p data is consistent with pure S wave scattering [20] [21]. Therefore, we assume the p-p scattering amplitude is

$$\left(\tfrac{1}{2}\underline{k}_0 + \underline{p}_f \middle| \{1 - P_{23} Q_{23}\} R_{pp}(23) \middle| \tfrac{1}{2}\underline{k}_0 + \underline{p}_i \right)$$

$$= \frac{e^{i\delta} \sin\delta}{\left| \tfrac{1}{2}\underline{k}_0 + \underline{p}_i \right|} \; \tfrac{i}{2} \left(1 - \underline{\sigma}_2 \cdot \underline{\sigma}_3\right) \qquad (4.8)$$

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[20] Chamberlain, Segre, and Wiegand, Phys. Rev. 81, 661 (1951).

[21] Bleke, Kruse, and Ramsey, Phys. Rev. 83, 274 (1951).

The phase shift δ is adjusted to yield the observed p-p cross section.

To calculate the n-p scattering amplitude, we assume a potential and use the Born approximation. A criterion for choosing among the various proposed potentials is that the Born approximation to the scattering amplitude must contain terms of the form needed to detect the polarization considered in Sec. III. A familiar potential which satisfies this requirement is a mixture of central and tensor forces. Assuming the Yukawa radial dependence, we use parameters which have been adjusted to fit low energy data.[22] Also the even exchange dependence is assumed for both central and tensor potentials. This model is almost identical to one for which the 90 Mev scattering cross-section has been calculated by a phase shift method.[23] Since this is the energy region of interest, we may partially check the validity of the Born approximation by comparing its results with more accurate ones. The resulting n-p scattering amplitude is

$$(\underline{q}'|R_{np}(12)|\underline{q}) = \frac{M V_0 \lambda_c^3}{2\pi}\left[\{u(\xi)+u(\eta)\}\right.$$
$$\left. + P\left(\frac{\lambda_c}{\lambda_t}\right)^2\{V(\xi)(3\underline{\sigma}_1\cdot\hat{\xi}\,\underline{\sigma}_2\cdot\hat{\xi}-\underline{\sigma}_1\cdot\underline{\sigma}_2)+V(\eta)(3\underline{\sigma}_1\cdot\hat{\eta}\,\underline{\sigma}_2\cdot\hat{\eta}-\underline{\sigma}_1\cdot\underline{\sigma}_2)\}\right] \quad (4.9)$$

Here
$$\underline{\xi} = \underline{q}' - \underline{q} \quad ; \quad \underline{\eta} = \underline{q}' + \underline{q} \quad (4.10)$$

$$u(\xi) = \frac{1}{1+(\lambda_c\xi)^2} \quad (4.11)$$

$$V(\xi) = \frac{1}{1+(\lambda_t\xi)^2} - \frac{3}{(\lambda_t\xi)^2}\left(1-\frac{\arctan\lambda_t\xi}{\lambda_t\xi}\right) \quad (4.12)$$

The usual notation for the parameters in the potential is used.

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(22) Bease and Feshbach, Phys. Rev. 81, 142 (1951. We use the set which fits the full binding energy of the triton, as the range of the tensor force then agrees with a model of reference 18. However g has been set equal to zero.

(23) Christian and Hart, Phys. Rev. 77, 441 (1950), particularly curve II, fig. 19

We have evaluated (4.9) for 90 Mev n-p collisions, and compared our results to those of Christian and Hart. The angular distributions are in fair agreement, but there is considerable disagreement between the total cross-sections, the Born method yielding a value which is only about two-thirds of the more accurate result. It is not clear which parts of the scattered amplitude are poorly approximated by the Born method. We have tried to gain some insight into the matter by the following argument, which is at best very rough. In the Born approximation, the central and tensor scattering amplitudes simply add. It has been shown that the first Born approximation does give an accurate cross-section at 90 Mev, presumably fortuitously.[24] This may be true only for the particular potential strength required when tensor forces are neglected, and not for the central force in our model, which is three-fourths as strong. But to make some comparison, we assume this is roughly valid for these reduced strengths, and then find the cross-section due to tensor forces to be about half of that required for agreement with the results of Christian and Hart. This may be an indication that the Born approximation is inaccurate in calculating the scattering by non-central forces for the potential and energy under consideration.

Although we do not believe the above calculation is very good, we may observe that (4.9) is qualitatively general for our purposes. The only terms which are allowed by invariance arguments, but do not appear in (4.9) are of the form $(\underline{\sigma}_1 + \underline{\sigma}_2) \cdot \hat{n}$ and $\underline{\sigma}_1 \cdot \underline{\sigma}_2$, where $\underline{n}$ is normal to the scattering plane. Terms of these types might appear in a more accurate calculation. It is readily shown that if such terms are added to (4.9), the only terms added to H^*H (calculated according to (4.5), (4.6), and (4.8)) are of the form

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(24) Jost and Pais, Phys. Rev. 82, 840 (1951)

$(\underline{\sigma}_1 + \underline{\sigma}_2)\cdot\underline{n}$ (or unity). Since we are interested in detecting polarisation specified by second rank tensor operators, the spin term is of no interest. If we were interested in detecting polarisation specified by expectation values of the spin, we should have to calculate the scattering amplitude more accurately, or use a different model. We shall therefore use (4.9) here despite its limitations.

Substituting (4.8) and (4.9) into (4.5) and (4.6), we may evaluate H and then H^*H. The result is

$$M^\dagger M = F(\underline{p} - \underline{p}_0)^2 \left[\frac{\sin^2\delta}{|\tfrac{1}{2}\underline{p}_0 + \underline{p}_1|^2} + A_0^2\{u(s) + u(r)\}^2 \right.$$

$$+ 6\tau^2 A_0^2 \{V(s)^2 + V(r)^2 - V(s)V(r)\}$$

$$+ \frac{\cos\delta \sin\delta}{|\tfrac{1}{2}\underline{p}_0 + \underline{p}_1|} A_0 \{u(s) + u(r) - \tau V(s)(3\underline{\sigma}_1\cdot\hat{s}\,\underline{\sigma}_2\cdot\hat{s} - \underline{\sigma}_1\cdot\underline{\sigma}_2)$$

$$\left. - \tau V(r)(3\underline{\sigma}_1\cdot\hat{r}\,\underline{\sigma}_2\cdot\hat{r} - \underline{\sigma}_1\cdot\underline{\sigma}_2)\} + \cdots \right] \qquad (4.13)$$

Here $r_0 = \frac{MV_0\lambda_c^2}{2\pi}\lambda_c$; $\tau = \Gamma\left(\frac{\lambda_c}{\lambda_\pi}\right)^3$ and the terms indicated by + ... are linear in $\underline{\sigma}_1$ and therefore will vanish when traces are taken.

Evaluating (4.13), the azimuthal angular distribution for polar angles of 30° and 60° are

$$I(30°,\phi) = I_0(30°)\left[1 + 0.18\langle T_{20}\rangle_i + 0.24\{\mathrm{Re}\langle T_{21}\rangle_i \cos\phi - \mathrm{Im}\langle T_{21}\rangle_i \sin\phi\} + 0.16\{\mathrm{Re}\langle T_{22}\rangle_i \cos 2\phi - \mathrm{Im}\langle T_{22}\rangle_i \sin 2\phi\}\right]$$

$$I(60°,\phi) = I_0(60°)\left[1 + 0.076\langle T_{20}\rangle_i - 0.051\{\mathrm{Re}\langle T_{21}\rangle_i \cos\phi - \mathrm{Im}\langle T_{21}\rangle_i \sin\phi\}\right. \qquad (4.14)$$

The small magnitude of the coefficient of $\langle T_{2,-1} \rangle_i$ in (4.15) is due to a cancellation of terms, each of which is of the same order as the coefficient of $\langle T_{2,-2} \rangle_i$. Since this cancellation depends on the particular models and the various approximations used, the coefficient of $\langle T_{2,-2} \rangle$ i. Since this cancellation depends on the particular models and the various approximations used, the coefficient of $\langle T_{2,-1} \rangle$ i in (4.15) may be of the same order as the coefficient of $\langle T_{2,-2} \rangle$ i.

Since only second rank tensors are of interest here, the discussion of Sec. I shows that the greatest asymmetries occur if the initial state is a pure state of the type χ_0 oriented favorably relative to the directions of motion. To study the efficiency of this detector, we calculate the asymmetries which it may produce for incident deuteron states of this type with varying orientation. We use $\hat{\beta}$ to represent a unit vector making an angle β with the z-axis and in the xz-plane, this is the assumed direction of orientation of the state. Then the non-vanishing expectation values are

$$\langle T_{LM} \rangle_i = -\sqrt{2}\, Y_{2m}(\hat{\beta}) \tag{4.16}$$

Substituting in (4.14) or (4.15) yields the result

$$W = 1 + C_1 \cos\phi + C_2 \cos 2\phi \tag{4.17}$$

Here W represents the relative intensity at various azimuthal angles. The functions $\sin\phi$ and $\sin 2\phi$ are absent from (4.17) because of our choice of xz-plane. Values of C_1 and C_2 for various angles β and both 30° and 60° scattering are listed in Table II.

Table II

	$\theta = 30^\circ$		$\theta = 60^\circ$	
β	C_1	C_2	C_1	C_2
0°	0	0	0	0
30°	-0.21	-0.040	+0.041	-0.079
45°	-0.22	-0.075	+0.046	-0.13
60°	-0.18	-0.10	+0.038	-0.18
90°	0	-0.13	0	-0.23

From Table II, we see that the asymmetries which may be produced with favorably oriented polarized deuterons with maximum value of the second rank tensor operators is not particularly large; calculated asymmetries in n-p scattering are of the same order.[25]

For deuterons produced in the $(p\,p,\pi^0 d)$ reaction whose polarization is given in Table I, the scattered intensity for polar angles 30° and 60° may be calculated directly from (4.14) and (4.15). The azimuthal dependence of this intensity is given by (4.17), with the constants tabulated in Table III. The parameters a and ϵ are the same as used in Sec. III.

Table III

		$\theta = 60^\circ$		$\theta = 30^\circ$	
a	ϵ	C_1	C_2	C_1	C_2
0.1	0.0	0.0047	-0.17	-0.032	-0.098
	0.5	0.0039	-0.084	-0.016	-0.049
	1.0	0.0010	+0.0004	-0.0045	+0.0002
0.2	0.0	0.0073	-0.19	-0.034	-0.11
	0.5	0.0052	-0.095	-0.024	-0.054
	1.0	0.0030	+0.0012	-0.014	+0.0007

(25) L. Wolfenstein, Phys. Rev. 76, 541 (1949).

Finally we should like to note some general features of the polarisation effects in d-p scattering. These do not vanish in the Born approximation, nor does our assumption that the proton-proton scattering amplitude contains only S-waves lead to vanishing results. Since such assumptions lead to vanishing polarisation effects in nucleon-nucleon scattering, the manner in which polarisation effects in d-p and nucleon-nucleon scattering depends on the fundamental scattering amplitudes is different even if the impulse approximation is valid. If the various experiments were feasible, they would tend to supplement each other.